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USE OF SENSITIVITY FUNCTIONS IN THE PROBLEM OF DESIGNING A MULTILAYER HEAT SHIELD

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An approach is proposed for solving the problem of designing a multilayer heat shield with a prescribed structure from the restrictions on its temperature.

The solution of a problem of the form

$$M = \sum_{j=1}^{n} \rho_{\text{var},j} h_{\text{var},j} \to \min_{\bar{h}_{\text{var}}}, \tag{1}$$

$$T_{\text{con},i} \leqslant \hat{T}_{\text{con},i}, \quad i = \overline{1, m}, \tag{2}$$

$$h_{\text{var},j} \geqslant \check{h}_{\text{var},j} \tag{3}$$

gives the weighted-optimal solution of the problem of designing a one-dimensional multilayered construction (packet) of a prescribed structure, which is exposed to a high-temperature medium and is characterized by restrictions on the temperature in separate zones of the structure. The temperature in the packet is described by the one-dimensional Fourier equation [1]

$$\rho c(T) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial y} \left(\lambda (T) \frac{\partial T}{\partial y} \right). \tag{4}$$

With the help of the method of the penalty functions [2] the starting problem (1)-(3) can be reduced to an unconditional-minimization problem

$$F = \sum_{j=1}^{n} \rho_{\text{var},j} h_{\text{var},j} + \sum_{i=1}^{m} a_i \max(0,\ T_{\text{con},i} - \hat{T}_{\text{con},i}) + \sum_{j=1}^{n} b_j \max(0,\ \check{h}_{\text{var},j} - h_{\text{var},j}) \to \min_{\bar{h}_{\text{var}}}. \tag{5}$$

The difficulties arising in the development of methods for solving problems of this kind are discussed in [3, 4]. However, these methods are not widely employed for investigating practical design questions. A simplified approach to the synthesis of structures, based on finding the combination of thicknesses of m separate layers such that conditions of the type

$$\varphi_i (h_{\text{var},1},\ \ldots,\ h_{\text{var},m}) = T_{\text{con},i} (h_{\text{var},1},\ \ldots,\ h_{\text{var},m}) - \hat{T}_{\text{con},i} = 0,\ i = \overline{1, m} \tag{6}$$

are satisfied, is employed much more often.

The present paper is devoted to methodological questions concerning the construction of the solution to problems of the type formulated above.

One possible algorithm for solving the problem (6) by iteration consists of the following sequence of operations which are performed at each k-th iteration:

formation of the initial approximation $h^{(k)}_{\text{var},j}$ $(j = \overline{1, m})$ for the unknown thicknesses of the layers;

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calculation of the values of the functionals $\varphi_i^{(k)}$ and their partial derivatives $\varphi_{h,i,j}^{(k)}$ with respect to the arguments $h_{var,j}$ ($i, j = \overline{1, m}$) using the heat-conduction equation;

determination of the increments to the thicknesses of the layers $\Delta h_{var,j}^{(k)}$, satisfying the system of linear algebraic equations

$$\sum_{j=1}^{m} \varphi_{h,i,j}^{(k)} \Delta h_{var,j}^{(k)} = -\varphi_i^{(k)}, \quad i = \overline{1, m}, \tag{7}$$

obtained by linearizing the system of equations (6).

The transition from the k-th iteration to the (k + 1)-st iteration is made with the help of formulas of the type

$$h_{var,j}^{(k+1)} = h_{var,j}^{(k)} + \Delta_{h,j}^{(k)}, \quad j = \overline{1, m}, \quad k = 1, 2, \ldots; \tag{8}$$

$$\Delta_{h,j}^{(k)} = \begin{cases} \Delta h_{var,j}^{(k)}, & |\Delta h_{var,j}^{(k)}| \leqslant |\Delta \hat{h}_{var,j}^{(k)}|, \\ \Delta \hat{h}_{var,j}^{(k)}, & |\Delta h_{var,j}^{(k)}| > |\Delta \hat{h}_{var,j}^{(k)}|, \quad j = \overline{1, m}; \end{cases} \tag{9}$$

$$\Delta \hat{h}_{var,j}^{(k)} = \beta h_{var,j} \operatorname{sign}(\Delta h_{var,j}^{(k)}), \quad j = \overline{1, m}. \tag{10}$$

The iteration process terminates at the ℓ-th iteration at which the conditions

$$|\varphi_i^{(l)}| \leqslant \varepsilon_i, \quad i = \overline{1, m} \tag{11}$$

are satisfied.

One of the basic elements of the solution of the problem using the algorithm presented above is the calculation of sensitivity functions. These functions are derivatives of the temperatures of the packet with respect to the thicknesses of the variable layers.

One possible method for calculating the functions $\varphi_{h,j}$ is based on their finite-difference approximation. This, naturally, requires repeated solution of a nonstationary heat-conduction problem. This approach is quite simple to implement. However, the time required to solve the problem increases rapidly with increasing number of variable layers in the packet, and there are often certain difficulties in choosing the step for the finite-difference approximation of the derivatives $\varphi_{h,j}$.

A different, alternative approach to the solution of the problem under study is implemented in the present paper. In this approach the sensitivity functions are determined directly from the differential heat-conduction equation (4).

To this end we introduce in the k-th layer of the packet the dimensionless coordinate $\bar{y} = (y - y_{0,k})/h_k$ and rewrite for it Eq. (4) in the form

$$\frac{1}{h_k^2} \frac{\partial}{\partial \bar{y}} \left(\lambda_k(T) \frac{\partial T}{\partial \bar{y}} \right) = \rho_k c_k(T) \frac{\partial T}{\partial \tau}. \tag{12}$$

Next, differentiating Eq. (12) with respect to the thickness of the variable layer $h_{var,j}$ and returning to the dimensional coordinate y, we obtain the equations for the sensitivity functions in the form

$$\rho_k c_k(T) \frac{\partial \varphi_{h,j}}{\partial \tau} = \frac{\partial}{\partial y} \left(\lambda_k(T) \frac{\partial \varphi_{h,j}}{\partial y} \right) - \left(\bar{c}_{T,k}(T) \varphi_{h,j} + 2 \frac{\delta_{j,k}}{h_k} \right) \times \\ \times \rho_k c_k(T) \frac{\partial T}{\partial \tau} + \frac{\partial}{\partial y} \left(\bar{\lambda}_{T,k}(T) \varphi_{h,j} \lambda_k(T) \frac{\partial T}{\partial y} \right), \quad j = \overline{1, m}, \tag{13}$$

where $\delta_{j,k} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k; \end{cases}$

$$\bar{\lambda}_{T,k}(T) = \lambda_{T,k}(T)/\lambda_k(T); \quad \bar{c}_{T,k}(T) = c_{T,k}(T)/c_k(T).$$

The boundary conditions on the surfaces w and v of the structural packet in the general case have the form

TABLE 1. Types of Packets

Type of packet	1				2					3				
Number of layers	4				5					5				
Layer number	1	2	3	4	1	2	3	4	5	1	2	3	4	5
Material number	1	2	3	4	2	5	0	3	4	2	5	3	5	2
Number of variable layers	3				2					2				
Number of layer varied	1	2	3		1			4		1				5
Thickness of fixed layer, mm				3		0,5	1		3		0,5	5	0,5	
Number of controlled junctions	3				2					2				
Number of controlled junction	1	2	3		2			4		2			5	
Maximum admissible temp. of controlled junction	1773	1273	343		1073			343		1073			1073	

TABLE 2. Properties of the Materials

Number of material	$\lambda(T)$	$c(T)$	ρ
1	$30-5\bar{T}$	$1500+0,2T$	2100
2	$(1+\bar{T}+\bar{T}^2)\,0,05$	$800+0,3T$	200
3	$0,04+0,01\bar{T}+0,03\bar{T}^2$	$1800-T+400\bar{T}^2$	100
4	$0,4+0,1\bar{T}$	$-400+5T$	1800
5	$20+25\bar{T}$	$150+0,5T$	7880

TABLE 3. Formulas for Calculating Heat Exchange at the Surface w of the Packet

Computational formulas			
q_w	$(\alpha/c_p)_w$	$I_{e,w}$	I_w
$(\alpha/c_p)_w(I_{e,w}-I_w)-\varepsilon_w\sigma T_w^4$	$0,03\exp\left[\left(3\frac{\tau-1000}{1000}-2\right)\left(\frac{\tau-500}{500}\right)^2\right]$	$3\cdot10^5(1+100\cos(\pi\tau/200))$	$954\,T_w+0,0862\,T_w^2$

TABLE 4. Formulas for Calculating Heat Exchange at the Surface v of the Packet

Packet type	Computational formulas			
	q_v	$(\alpha/c_p)_v$	$I_{e,v}$	I_v
1	$15(323-T_v)+$	—	—	—
2	$+\varepsilon_v\sigma(333^4-T_v^4)$	—	—	—
3	$(\alpha/c_p)_v(I_{e,v}-I_v)-\varepsilon_v\sigma T_v^4$	$0,2\,(\alpha/c_p)_w$	$I_{e,w}$	$954\,T_v+0,0862\,T_v^2$

TABLE 5. Results of Solving the Problem of Designing a Packet

Type of packet	Initial approx. $\bar{h}_{var}^{(0)}\cdot 10^3$	Problem solution $\bar{h}_{var}^{(*)}\cdot 10^3$	Scheme	
			1, τ_c	2, τ_c
1	(5; 5; 10)	(17,7; 20,1; 35,2)	290	74
2	(5; 10)	(25,3; 32,4)	399	159
3	(10; 5)	(39; 5,9)	649	99

$$-\frac{\lambda_1(T)}{h_1}\frac{\partial T}{\partial \bar{y}} = q_w(T); \tag{14}$$

$$\frac{\lambda_n(T)}{h_n}\frac{\partial T}{\partial \bar{y}} = q_v(T). \tag{15}$$

The specific form of the functions $q_w(T)$ and $q_v(T)$ is presented below for different cases.

The boundary conditions for the sensitivity functions are obtained by differentiating the relations (14) and (15) with respect to the thickness of the varied layer $h_{var,j}$:

$$-\lambda_1(T)\frac{\partial \varphi_{h,j}}{\partial y} = q_{w,T}(T)\,\varphi_{h,j} + \lambda_1(T)\frac{\partial T}{\partial y}\left(\bar{\lambda}_{T,1}(T) - \frac{\delta_{j,1}}{h_1}\right); \tag{16}$$

$$\lambda_n(T)\frac{\partial \varphi_{h,j}}{\partial y} = q_{v,T}(T)\,\varphi_{h,j} - \lambda_n(T)\frac{\partial T}{\partial y}\left(\lambda_{T,n}(T) - \frac{\delta_{j,n}}{h_n}\right). \tag{17}$$

The boundary conditions on the inner surfaces of the packet, which bound the gas interlayers, if, naturally, such layers are present, are constructed in a similar manner.

The numerical solutions of both the direct problem of heat conduction and the problem of determining the sensitivity functions are constructed in the present work using implicit difference schemes for Eqs. (4) and (13) and the sweep method [5]. Since the equations for the sensitivity functions are linear, these functions are determined at the last iteration of the solution of the direct heat-conduction problem.

In order to show that the proposed method is effective and works we performed an extensive numerical experiment. As an illustration of the investigations performed we examine the results of the solution of the design problem for three types of constructions, which are presented in Table 1; the material with the number zero corresponds to an air interlayer.

The thermophysical properties of the materials and the formulas employed for calculating the heat-transfer parameters are presented in Tables 2-4. The emissivities of all surfaces in the packet, which participate in radiation heat exchange, were assumed to be equal to 0.8, while the parameter β, appearing in the formula (10) and limiting the rate of change of the parameter $h_{var,j}$ in the iteration process, was set equal to 0.2. The initial temperature of the packets was equal to 300 K, and the process studied was assumed to last for 1000 sec.

The main results of the investigation, performed using an El'brus 1-2K computer, are presented in Table 5. Implementation of the algorithm according to the first scheme corresponds to determining the sensitivity functions with the help of a finite-difference approximation, while second scheme corresponds to solving the corresponding differential equations.

As one can see from Table 5, the second scheme for determining the sensitivity functions gives a much more efficient algorithm for solving the problem of designing heat shield.

NOTATION

M, F, and φ, functionals; T, temperature, K; y, spatial coordinate, m; $y_{0,k}$, coordinate of the start of the k-th layer, measured from the surface w, m; $\bar{T}_{con,i}$, temperature of the i-th controlled junction, K; $\hat{T}_{con,i}$, maximum admissible temperature of the i-th controlled junction, K; $y_{con.i}$, coordinate of the i-th controlled junction, m; $\bar{T} = T/1000$; $\varphi_{h,j} = \partial T/\partial h_{var,j}$; $\varphi_{h,i,j} = \varphi_{h,j}(y_{con.i})$; q_w and q_v, heat fluxes supplied to the boundaries w and v of the packet, W/m²; $q_{w,T} = \partial q_w/\partial T$; c, specific heat capacity, J/(kg·K); λ, thermal conductivity; W/(m·K); ρ, density of the material, kg/m³; $\bar{h}_{var} = (h_{var,1}, \ldots, h_{var,m})$, collection of variable layers in the packet, m; $\check{h}_{var,j}$, minimum admissible thickness of the j-th variable layer, m; ε_w and ε_v, the emissivities of the boundary surfaces w and v of the packet; $I_{e,w}$ and $I_{e,v}$, enthalpies of restoration of gas flow at the boundary surfaces of the packet, J/kg; I_w and I_v, enthalpies of restoration of gas flow at the temperatures of the walls w and v, J/kg; $(\alpha/c_p)_w$, $(\alpha/c_p)_v$, heat-transfer coefficients of the surfaces w and v, $kg/(m^2 \cdot sec)$; σ, Stefan–Boltzmann constant, $W/(m^2 \cdot K^4)$; $\varepsilon_i = \varepsilon = 0.01$, required accuracy of the iteration process; τ, time, sec; τ_c, computing time, sec; and a_i and b_j, coefficients in the penalty function.

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